COO-3077-32
MF-73

Courant Institute of
Mathematical Sciences

Magneto-Fluid Dynamics Division

Instabilities and Growth Rate of Guiding Center Diffuse Pinches

Young-ping Pao

AEC Research and Development Report

Plasma Physics
August 1973

New York University

UNCLASSIFIED

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Contract No. AT(11-1)-3077

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Preface

This report contains two parts. Part 1 derives the normal mode stability equation for a diffuse pinch of guiding center plasma and examines the guiding center Suydam instability and its growth rates. Part 2 studies the pinch stability problem in the low shear limit.

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Part 1

STABILITY EQUATION AND LOCAL INSTABILITY

I. Introduction

A convenient approach in studying plasma stability problems is a δW variational analysis. This yields a δW stability criteria but gives no information on the growth rate. An alternative approach is to use the normal mode analysis for the full set of equations of motion, or, equivalently the variational analysis including the kinetic energy. This not only gives the stability criteria but also yields information regarding the growth rate and the spectrum of the problem.

The present work is concerned with the stability problem of a cylindrical linear diffuse pinch of collisionless guiding center plasma. For this problem, the δW variational analysis has been studied by Aleksin and Yashin.[1] The purpose of the present work is to study this problem by using the normal mode analysis based on the full set of equations of motion of the lowest order guiding center theory[2,3,4] and obtain information regarding the growth rate and the spectrum of the problem. It is found that the full set of equations of motion can be reduced to a single second order ordinary differential equation [see Eq. (57)] which is convenient for both analytic and numerical purposes. This second order stability equation, which contains the growth rate ω in it and which reduces to the equation of Aleksin and Yashin in the special limit $\omega = 0$, can be viewed as the guiding center version of the Hain-Lüst equation for the MHD diffuse pinch[5]. It is somewhat similar

to the Hain-Lüst equation except that its coefficients contain Cauchy-type integrals of the equilibrium distribution functions.

In a first application, this equation is then used to study the Suydam-type instability and the associated spectrum. The result confirms the MHD result of Grad[6] that Suydam instability has infinitely many discrete unstable eigenvalues with the origin as accumulation point. And it is shown that the eigenvalues ω_n for large n satisfy the linear relationship

$$\omega_{n+1}/\omega_n = \exp(-\pi/2c)$$

where c is a constant parameter [see eq. (74)]; this result has been born out by the numerical calculation of Goedbloed and Sakanaka[7] for the MHD diffuse pinch.

Further stability study, including line-preserving and semi-local instabilities[6], based on the stability equation [eq. (57)] is currently in progress.

It is assumed that in the equilibrium state the plasma properties and the magnetic field vary with the radial variable r, but are independent of the azimuthal variable ϕ and the axial variable z. The equations of motion are studied in Section II and it is shown in Section III that the result can be reduced to a second order ordinary differential equation. The boundary conditions are formulated in Section IV by assuming that the plasma is either bounded by a conducting wall or is separated from the wall by a vacuum region. In Section V, the eigenvalue problem associated with the Suydam-type instability is studied and the spectrum is also discussed.

II Basic Equations

Consider a plasma consisting of electrons and one kind of positive ion with distribution functions $F^{\pm}$. We shall use the version of guiding center theory given in Reference 2 and write $F^{\pm} = F^{\pm}(\mu, v, \underset{\sim}{x}, t)$ where μ is the magnetic moment and v is the particle velocity along the magnetic field line. In terms of $F^{\pm}$, the density $\rho^{\pm}$ and the pressure components $P_1^{\pm}$ and $P_2^{\pm}$ are

$$[\rho, P_1, P_2]^{\pm} = \int [1, v^2, \mu B] F^{\pm} \, d\mu dv \tag{1}$$

and for the plasma as a whole we have $\rho = \rho^+ + \rho^-$, $P_1 = P_1^+ + P_1^-$ and $P_2 = P_2^+ + P_2^-$. The pressure tensor is then

$$\underset{\sim}{P} = \Delta \underset{\sim}{B}\underset{\sim}{B} + P_2 \underset{\sim}{I} \tag{2}$$

where $\Delta = (P_1 - P_2)/B^2$ and $\underset{\sim}{I}$ is the unit tensor.

Let $\underset{\sim}{U}$ be the macroscopic plasma velocity and introduce $\underset{\sim}{u} = \underset{\sim}{U} - (\underset{\sim}{U} \cdot \underset{\sim}{\beta})\underset{\sim}{\beta}$ where $\underset{\sim}{\beta}$ is the unit vector $\underset{\sim}{B}/B$. The basic equations are

$$\rho \frac{d\underset{\sim}{U}}{dt} = -\nabla \underset{\sim}{P} + (\nabla \times \underset{\sim}{B}) \times \underset{\sim}{B} \, , \tag{3}$$

$$\frac{\partial \underset{\sim}{B}}{\partial t} = \nabla \times (\underset{\sim}{u} \times \underset{\sim}{B}) \, , \tag{4}$$

$$\nabla \cdot \underset{\sim}{B} = 0 \, . \tag{5}$$

On the other hand the motion of guiding centers along the

magnetic field lines is governed by the one-dimensional kinetic equation

$$0 = \frac{\partial F^{\pm}}{\partial t} + \nabla\cdot[(\underset{\sim}{u} + v\underset{\sim}{\beta})F^{\pm}] + \underset{\sim}{\kappa}\cdot\underset{\sim}{u}(vF^{\pm})_v + [\alpha^{\pm}T + \underset{\sim}{\beta}\cdot\nabla(\tfrac{1}{2}u^2 - \mu B)]F^{\pm}_v \quad (6)$$

where the subscript v denotes differentiation with respect to v, $\underset{\sim}{\kappa}$ is the magnetic field curvature,

$$T = \underset{\sim}{\beta}\cdot[(\nabla\underset{\sim}{P}/\rho)^{+} - (\nabla\underset{\sim}{P}/\rho)^{-}] \quad (7)$$

and the constants $\alpha^{\pm}$ is given by

$$\alpha^{\pm} = (e/M)^{\pm}/[(e/M)^{+} - (e/M)^{-}] \ , \quad (8)$$

where $e^{\pm}$ and $M^{\pm}$ are respectively the charge (signed) and the mass of the ions and electrons.

Following the normal-mode method, we consider the perturbations

$$F = F^{o} + f\ \exp(i\theta) \ , \quad (9)$$

$$P_{1,2} = P^{o}_{1,2} + p_{1,2}\ \exp(i\theta) \ , \quad (10)$$

$$\underset{\sim}{B} = \underset{\sim}{B}^{o} + \underset{\sim}{b}\ \exp(i\theta), \quad (11)$$

$$\underset{\sim}{U} = \underset{\sim}{U}\ \exp(i\theta), \quad \rho = \rho^{o} + \rho_1\ \exp(i\theta) \ , \quad (12)$$

where $\theta = (-\omega t + kz + m\phi)$ in the cylindrical coordinates (r, ϕ, z) and the superscript o denotes equilibrium values. Only unstable modes are considered and it is therefore assumed throughout that w is not real. For the equilibrium state, the pressure balance requires that

$$\dot{P}_* + r^{-1}(\sigma B_\phi^o)^2 = 0 \tag{13}$$

where $P_* = P_2^o + \frac{1}{2}(B^o)^2$, the dot denotes differentiation with respect to r, σ is given by

$$\sigma^2 = 1 - \Delta^o \tag{14}$$

and Δ^o is defined in (2). It is further assumed that for the equilibrium state both σ^2 and the quantity $K_1(0)$ given by (38)

$$\sigma^2 > 0, \quad K_1(0) > 0 \tag{15}$$

so that the fire hose and mirror instabilities[8] associated with $\sigma^2 < 0$ and $K_1(0) < 0$ will not enter the present pinch problem.

For convenience of writing, we shall introduce the following notation

$$g = \underset{\sim}{B}^o \cdot \underset{\sim}{b}, \quad b_n = (B_z^o b_\phi - B_\phi^o b_z)/B^o \tag{16}$$

$$a = \text{div}\, \underset{\sim}{u}, \quad u_n = (B_z^o u_\phi - B_\phi^o u_z)/B^o \tag{17}$$

$$\tau = p_2 + g, \quad \zeta = p_1 - p_2 \tag{18}$$

$$H = kB_z^o + (m/r)B_\phi^o, \quad \nu = B_\phi^o/(rB_z^o) \;, \tag{19}$$

$$E = kB_\phi^o - (m/r)B_z^o \;, \tag{20}$$

and denote by $\hat{r}$ and $\hat{\Phi}$ the unit vectors in the r and Φ directions respectively.

Now substituting (10)-(12) into the equation of motion (3) yield the linearized equation

$$-\rho\, i\omega \underset{\sim}{U} = -\nabla\tau - B^{-2}(\zeta - 2\Delta g)(iH\underset{\sim}{B} - r^{-1}B_\phi^2\hat{r}) + b_r[\sigma^2(\dot{\underset{\sim}{B}} + r^{-1}B_\phi\hat{\Phi}) - \dot{\Delta}\underset{\sim}{B}]$$
$$+ \sigma^2[iH\underset{\sim}{b} - 2r^{-1}B_\phi b_\phi\hat{r}] \tag{21}$$

where we have omitted the superscript o from the equilibrium values ρ^o, B^o and Δ^o for simplicity. Similarly, equations (4) and (5) yield

$$i\omega\underset{\sim}{b} = -\;H\underset{\sim}{u} + a\underset{\sim}{B} + u_r(\dot{\underset{\sim}{B}} - r^{-1}B_\phi\hat{\Phi}) \;, \tag{22}$$

$$\dot{b}_r + (b_r/r) + ikb_z + i(m/r)b_\phi = 0 \;, \tag{23}$$

where (23) is not independent of (22).

Now substituting (9)-(12) into (6) gives the following linearized kinetic equation

$$f^{\pm} = iH \quad B^{-1}F_v^{\pm}[\alpha^{\pm}\eta - B^{-2}(\alpha^{\pm}V + B\mu)g]\delta + B^{-2}gF^{\pm} + (i\omega)^{-1}(\dot{F}^{\pm} - B^{-1}\dot{B}F^{\pm})u_r$$
$$- u_r F_v^{\pm}[(rB^2)^{-1}B_\phi^2 v + H(B\omega)^{-1}(\alpha^{\pm}S - \mu\dot{B})]\delta \ , \tag{24}$$

where again we have omitted the superscript o from F and $\underset{\sim}{B}$. The function η, δ, V and S are

$$\eta = (p_1/\rho)^+ - (p_1/\rho)^- \ , \tag{25}$$

$$\delta = (i\omega - iHB^{-1}v)^{-1} \tag{26}$$

$$V = [(P_1 - P_2)/\rho]^+ - [(P_1 - P_2)/\rho]^- \ , \tag{27}$$

$$S = (\dot{P}_1/\rho)^+ - (\dot{P}_1/\rho)^- - V(\dot{B}/B) \ , \tag{28}$$

Where it is recalled that ρ is the equilibrium plasma density, and it is noted that δ is always finite since only non-real values of w are considered. The perturbation $p_1^{\pm}$ and $p_2^{\pm}$ are related to the perturbed distribution function $f^{\pm}$ by

$$p_1^{\pm} = \int v^2 f^{\pm} \, d\mu dv \ , \tag{29}$$

$$p_2^{\pm} = B\int \mu f^{\pm} \, d\mu dv + B^{-2}P_2^{\pm}g \ , \tag{30}$$

$$p_1 = p_1^+ + p_1^- \ , \quad p_2 = p_2^+ + p_2^- \ . \tag{31}$$

The basic equations are eqns. (21)-(24) while eqns. (16)-(20) and eqns. (25)-(31) furnish the definitions of the various quantities that are involved in the basic equations.

III The Stability Equation

It is possible to reduce the basic equations (21)-(24) in the last section to a single second order ordinary differential equation. This will be done in the present section. The derivation involves considerable arithmetic manipulations and a complete description of the derivation will not be presented. In what follows only the major steps in the derivation will be given.

First substitute expression (24) for $f^{\pm}$ into (29) and (30) and use (25) and (31). This then gives rise to three equations connecting η, p_1, p_2, g and u_r. We can then solve these equations for η, p_1 and p_2 in terms of g and u_r. This result then enables us to express p_1 and p_2 in terms of linear combinations of g and u_r. If these expressions are used in (18) for τ and ζ, we can then write

$$\tau = K_1 g + (i\omega)^{-1} K_2 u_r \ , \tag{32}$$

$$\zeta = L_1 g + (i\omega)^{-1} L_2 u_r \ . \tag{33}$$

The coefficients K_1, K_2, L_1 and L_2 can be expressed in a convenient form if we introduce the functions

$$I(\ell, n) = \int_{\infty}^{\infty} \frac{dv}{v - w} \int_{0}^{\infty} \mu^{\ell} \left[\left(\frac{F_v}{\rho^n}\right)^{+} + (-1)^n \left(\frac{F_v}{\rho^n}\right)^{-} \right] d\mu \ . \tag{34}$$

$$\mathcal{L}_1(w) = I(2,0) - [I(1,1)I(1,1)/I(0,2)] \tag{35}$$

$$\mathcal{L}_2(w) = I(1,0) - [I(0,1)I(1,1)/I(0,2)] \tag{36}$$

$$\ell_3(w) = I(0,0) - [I(0,1)I(0,1)/I(0,2)] \tag{37}$$

$$w = \omega B/H \tag{38}$$

In terms of ℓ_1, ℓ_2 and ℓ_3, the coefficients in eqns. (32) and (33) can be written as

$$K_1(w) = 2B^{-2}P_2 + 1 + \ell_1(w) , \tag{39}$$

$$K_2(w) = -B\dot{B}K_1(w) - r^{-1}B_\phi^2[\sigma^2 - B^{-1}w^2\ell_2(w)] , \tag{40}$$

$$L_1(w) = \sigma^2 + 2\Delta - K_1(w) + w^2B^{-1}\ell_2(w) , \tag{41}$$

$$L_2(w) = \dot{P}_1 - B\dot{B}\Delta - K_2(w) - w^2\dot{B}\ell_2(w) - r^{-1}(B^\phi/B)^2[\rho w^2 - w^4\ell_3(w)]. \tag{42}$$

The main result from the one-dimensional guiding center kinetic equation is given by the expressions (32) and (33), which is all that is needed in the equations of motion. In this sense, the guiding center equation along the field plays the role of constitutive equations which furnish the relation between τ, ζ and the basic variables g and u_r.

We now turn to the equations of motion. The r-component of (21) is simply

$$-\rho i\omega u_r = -\dot{\tau} + r^{-1}(B_\phi/B)^2(\zeta - 2\Delta g) + \sigma^2[iHb_r - 2r^{-1}B_\phi b_\phi] . \tag{43}$$

In the direction perpendicular to both $\hat{r}$ and $\underset{\sim}{B}$, $\underset{\sim}{U}$ and $\underset{\sim}{u}$ have the same component u_n. The equation for u_n can be derived from (21) and the result is

$$-\rho i\omega B u_n = iE\tau + i\sigma^2 H b_n + b_r \sigma^2 B_z B_\phi[(\dot{\nu}/\nu) + (2/r)] . \tag{44}$$

where ν is given in (19).

On the other hand, the divergence of $\underset{\sim}{u}$ can be written in terms of u_r and u_n as

$$a = \dot{u}_r + r^{-1}u_r - iB^{-1}E\, u_n \tag{45}$$

and from the magnetic field equations (22) and (23), it follows that

$$iEBu_n = -i\omega g + (B\dot{B} + r^{-1}B_z^2)u_r + B^2\dot{u}_r , \tag{46}$$

$$i\omega b_n = B_z B_\phi(\dot{\nu}/\nu)u_r - iHBu_n , \tag{47}$$

$$i\omega b_\phi = aB_\phi + (\dot{B}_\phi - r^{-1}B_\phi)u_r - iHB_z B^{-1}u_n , \tag{48}$$

$$b_r = -H\omega^{-1}u_r , \tag{49}$$

from which it is readily seen that u_n, b_n, b_ϕ and b_r can all be expressed in terms of $u_r, \dot{u}_r$ and g. Substituting these expressions into (44) and using (32), we obtain

$$\Omega i\omega g = NB^2\dot{u}_r + Q\, u_r \tag{50}$$

where N, Ω and Q are given by

$$N = \rho\omega^2 - \sigma^2 H^2 , \tag{51}$$

$$Q = N(B\dot{B} + r^{-1}B_z^2) + E^2K_2 - 2r^{-1}\ \sigma^2 B_\phi B_z EH\ , \tag{52}$$

$$\Omega = N - E^2 K_1\ . \tag{53}$$

Finally, we substitute (32), (33), (48) and (49) into (43), using (46) and (50). The result is a second order ordinary differential equation

$$\frac{d}{dr}\left[\frac{K_1 NB^2}{\Omega}\ \frac{du_r}{dr}\right] + C_1(w,r)\ \frac{du_r}{dr} + C_o(w,r)\ u_r = 0 \tag{54}$$

where the coefficients C_o and C_1 are

$$\begin{aligned} C_1(w,r) &= \Omega^{-1}(K_1 Q + K_2\Omega) - (r\Omega)^{-1}(L_1 - 2\Delta)NB_\phi^2 \\ &\quad + 2(r\Omega)^{-1}\ \sigma^2 kEB^2K_1 + 2r^{-1}\ \sigma^2 B_\phi^2\ , \end{aligned} \tag{55}$$

$$\begin{aligned} C_o(w,r) &= N + \frac{d}{dr}[\Omega^{-1}(K_1 Q + \Omega K_2)] + 2r^{-1}\sigma^2 B_\phi \dot{B}_\phi \\ &\quad + 2(rE)^{-1}\sigma^2 kB_\phi[(Q/\Omega) - B\dot{B} - r^{-1}B_z^2] \\ &\quad - r^{-1}(B_\phi/B)^2\ [L_2 + (Q/\Omega)\ (\sigma^2 - K_1)]\ . \end{aligned} \tag{56}$$

A further simplification for (54) is possible. For this purpose, we substitute (39)-(41) into (55) and after some manipulations it is found that

$$C_1 = K_1 NB^2(r\Omega)^{-1}$$

and as a consequence, eqn. (54) may be written as

$$\frac{d}{dr}\left[\frac{rK_1 NB^2}{\Omega}\,\frac{du_r}{dr}\right] + C_o(w,r)\,ru_r = 0\ , \tag{57}$$

where C_o is given in (56). Eqn. (57) is the stability equation which forms the basis for all subsequent discussions.

In the limiting case for $w = 0$, eqn. (57) agrees with the variational result obtained by Aleksin and Yashin[1].

It is also noted that if $F^{\pm}$ are symmetric in v and non-increasing in v^2, then $K_1(0) > 0$ implies that $K_1(W) > 0$ for all imaginary W. It then follows that (57) is non-singular.

IV Boundary Conditions

If the plasma fills the cylinder up to the conducting wall at $r = R_o$, then the boundary condition for (56) at $r = R_o$ is simply

$$u_r\,(R_o) = 0 \,. \tag{58}$$

On the other hand if the plasma in equilibrium has a vacuum-plasma interface at $r = R_o$, then the interface must also be perturbed as

$$r = R_o + \xi \exp(i\theta)$$

and the kinematic boundary condition at the interface is

$$i\omega\xi + u_r = 0 \,. \tag{59}$$

Outside the plasma, the vacuum magnetic field may be denoted by

$$\underset{\sim}{B}_e = \underset{\sim}{B}_e^o + \underset{\sim}{b}_e \tag{60}$$

and the perturbed external field b_e can be expressed as

$$\underset{\sim}{b}_e = \nabla\tilde{\chi} \tag{61}$$

where $\tilde{\chi} = \chi(r)\ \exp(i\theta)$ and $\chi(r)$ is a linear combination of the modified Bessel functions of order m.

At the interface, continuity of the normal component of the magnetic field requires

$$b_r - i(H_e - H)\xi = -\dot{\chi}(r) , \tag{62}$$

where $H_e = kB_{ze} + (m/r)B_{\phi e}$, and we have dropped the superscript o from $\underset{\sim}{B}_e^o$. The pressure balance gives

$$\xi[-\ \sigma^2 B_\phi^2 - B_{\phi e}\dot{B}_{\phi e}] + \tau = H_e \chi . \tag{63}$$

If we substitute (32) for τ, (48) for b_r and (59) for ξ into (62) and (63), we find that

$$-\omega\dot{\chi} = (H_e - 2H)u_r , \tag{64}$$

$$-\omega H_e \chi = (K_3 - \frac{B_{\phi e}^2}{r})u_r + \Omega^{-1}K_1 NB^2 \dot{u}_r , \tag{65}$$

where K_3 is

$$K_3 = (K_1 Q)/\Omega + K_2 + \sigma^2 B_\phi^2 . \tag{66}$$

Equation (64) and (65) can be rearranged to give the final boundary condition

$$0 = \Omega^{-1} K_1 N B^2 \dot{u}_r + [K_3 - \frac{B^2_{\phi e}}{r} + (2H - H_e) H_e (\chi/\dot{\chi})] u_r \tag{67}$$

at $r = R_o$, where $\chi/\dot{\chi}$ is completely determined by the position of the conducting wall at $r = R_1$ which bounds the vacuum region.

Finally, it is noted that $r = 0$ is a regular singular point of (57). It is not difficult to show by displaying the indicial equation that as $r \to 0$, one solution of (57) tends to zero while any other linear independent solution diverges in such a way that

$$\int (\dot{u}_r)^2 r \, dr$$

is infinite which is unacceptable. Therefore the boundary condition at $r = 0$ may be taken as

$$u_r(0) = 0. \quad (m \neq 0) \tag{68}$$

V Suydam Instability and the Associated Eigenvalue Problem

In this section we consider the case in which H vanishes at a point $r = r_o$ in the plasma, and we shall be concerned with small values of w. The eigenvalue problem to be considered consists of the second order ODE in (57) and the boundary conditions which may be either (67) and (68), or (58) and (68).

To facilitate our discussions, it is expedient to consider three intervals $(o, r_o - \varepsilon)$, $(r_o - \varepsilon_1, r_o + \varepsilon_1)$ and $(r_o + \varepsilon, R_o)$ where $\varepsilon \ll \varepsilon_1 \ll r_o$. In the two side intervals $(o, r_o - \varepsilon)$ and $(r_o + \varepsilon, R_o)$, H is bounded away from zero so that ρw^2 is negligible compared to $\sigma^2 H^2$ and eqn. (57) may be simplified as

$$\frac{d}{dr}\left[\frac{rK_1(0)\sigma^2 H^2 B^2}{\sigma^2 H^2 + E^2 K_1(0)} \frac{du_r}{dr}\right] + rC_o(0,r)u_r = 0 \; . \tag{69}$$

As r in the two side intervals $(o, r_o - \varepsilon)$ and $(r_o + \varepsilon, R_o)$ tends to $r_o \pm \varepsilon$, we have

$$H^2 \cong m^2 B_z^2 \dot{\nu}^2 x^2 \tag{70}$$

$$\nu \cong -k/m \tag{71}$$

where $x = (r - r_o)$ and after some computations we find that

$$C_o(0, r_o) = -\frac{k^2 B^2}{r_o E^2}\left[\dot{P}_1 + \dot{P}_2 + \frac{\sigma^2 B_\phi^2}{r_o}\left(1 - \frac{\sigma^2}{K_1(0)}\right)\right] \tag{72}$$

Using the variable $x = r - r_o$, we can write eqn. (69) as

$$\frac{d^2u_r}{dx^2} + \frac{2}{x}\frac{du_r}{dx} + \frac{q}{x^2}u_r = 0 \tag{73}$$

where the coefficient q is $C_o(0,r_o)$ divided by $(\sigma HB/E)^2$. The solution of this equation is oscillatory if $q > \frac{1}{4}$, given by

$$|x|^{-1/2}\sin\Theta; \quad |x|^{-1/2}\cos\Theta$$

where $\Theta = c \log x^2$ and c is the index

$$c = \frac{1}{2}(q - \frac{1}{4})^{1/2} . \tag{74}$$

It may be claimed that a sufficient condition for instability is that $q > \frac{1}{4}$, which may be written in full as

$$\dot{P}_1 + \dot{P}_2 + (B_\phi^2/r)\sigma^2[1 - \frac{\sigma^2}{K_1(0)}] + \frac{1}{4}\sigma^2 B_z^2(\dot{\nu}/\nu)^2 < 0 . \tag{75}$$

This is the guiding center version of the Suydam condition for instability. It may be noted that if the electrons and the ions have the same distribution function and if the distribution function is a two-temperature Maxwellian, then $K_1(0) = 1$. If, moreover, the pressure is isotropic, then $\sigma^2 = 1$. Under these circumstances, (75) reduces to Suydam's result[9].

Condition (75) has previously been obtained by Aleksin and Yashin[1] using a variational analysis.

To show that (75) indeed implies instability and to

examine the nature of the eigenvalue problem, we now turn to the center layer $r_o - \varepsilon_1 \leq r \leq r_o + \varepsilon_1$.

We only need to deal with purely imaginary ω and can therefore write

$$\rho(r_o)\omega^2 E^{-2} = -\lambda^2 \ . \tag{76}$$

In the center layer, we return to the original equation, (57), and introduce a new variable

$$\tilde{x} = x/\lambda$$

in terms of which eqn. (57) may be written in the form

$$\frac{d}{d\tilde{x}}[\Gamma(\tilde{x}^2)\frac{d\tilde{u}_r}{d\tilde{x}}] + r_o C_o(\tilde{x}^2, r_o)\tilde{u}_r = 0 \tag{77}$$

where the tilde in $\tilde{u}_r$ indicates that the solution is only valid in the center layer $r_o - \varepsilon_1 \leq r \leq r_o + \varepsilon_1$. The function $\Gamma(\tilde{x}^2)$ is

$$\Gamma(\tilde{x}^2) = r_o(B/E)^2[1 + m^2 B_z^2 \dot{\nu}^2 \tilde{x}^2]$$

where B, E, B_z and $\dot{\nu}$ are evaluated at $r = r_o$.

The advantage of using $\tilde{x}$ is that eqn. (77) is now independent of w. As $\tilde{x} \to \pm \infty$, eqn. (77) has the same form as eqn. (73) and therefore the solutions of (77) have the asymptotic behavior

$$|\tilde{x}|^{-1/2} \sin \tilde{\Theta}; \quad |\tilde{x}|^{-1/2} \cos \tilde{\Theta} \tag{78}$$

as $\tilde{x} \to \pm \infty$, where $\tilde{\Theta} = c \log \tilde{x}^2$.

Since the coeffifients of (77) are symmetric in $\tilde{x}$, we can speak of a symmetric solution u_s and an anti-symmetric solution u_a. According to (78), we may express the asymptotic behavior of u_s and u_a as

$$u_s = |\tilde{x}|^{-1/2} \sin(\tilde{\Theta} + \Phi), \quad \tilde{x} \to \pm \infty, \tag{79}$$

$$u_a = \pm|\tilde{x}|^{-1/2}[\cos(\tilde{\Theta} + \Phi) + A \sin(\tilde{\Theta} + \Phi)], \quad \tilde{x} \to \pm \infty \tag{80}$$

where the constants Φ and A are independent of ω.

Now it is possible to put the pieces together. First we start with a solution u_r of (57) which satisfies the boundary condition (58) or (67) at $r = R_o$. As r moves from R_o to $r_o + \varepsilon$, this solution becomes

$$u_r \cong |x|^{-1/2} \sin(\Theta + \Phi_+) \quad . \tag{81}$$

And on the other hand, in the interval $(0, r_o - \varepsilon)$, we start with the boundary condition (68), and as $r \to r_o - \varepsilon$, we have

$$u_r \cong G|x|^{-1/2} \sin(\Theta + \Phi_-) \quad . \tag{82}$$

It is noted that $\Phi_{\pm}$ are also independent of ω.

What remains to be shown is that it is always possible to find a real λ and a solution $\tilde{u}_r$ of (77) such that $\tilde{u}_r$ matches

(81) as $\tilde{x} \to +\infty$ and (82) as $\tilde{x} \to -\infty$.

To illustrate the basic ideas in the proof, let us first consider the special case in which $\Phi_+ = \Phi_-$ (mod. 2π). In this case, we only need to take

$$\tilde{u}_r(\tilde{x}) = \lambda^{-1/2} u_s$$

which goes to

$$\tilde{u}_r \to |\tilde{x}\lambda|^{-1/2} \sin(\Theta - c \log \lambda^2 + \Phi)$$

as $\tilde{x} \to \pm\infty$. In order for this solution to match (81) and (82), it is only necessary to choose λ such that

$$c \log \lambda^2 = \Phi - \Phi_\pm - \pi n$$

where n is any integer. We therefore have

$$\begin{aligned} \lambda^2 &= -\rho(r_o)\omega^2 E^{-2} \\ &= \exp[c^{-1}(\Phi - \Phi_\pm - \pi n)] \ . \end{aligned} \tag{83}$$

If $\Phi_+ = \Phi_- + \pi$, the same argument goes through by taking G = -1.

It is seen from (83) that if n is large, then λ will be small which justifies our consideration that w is small.

Now we proceed to the general case in which $\Phi_+ \neq \Phi_-$ (mod. π). We start with (81) and in terms of the variable $\tilde{\Theta}$, eqn. (81) becomes

$$u_r \cong |\tilde{x}\lambda|^{-1/2}[\cos\psi_+ \sin(\tilde{\Theta}+\Phi) + \sin\psi_+ \cos(\tilde{\Theta}+\Phi)] , \tag{84}$$

where the constant ψ_+ is given by

$$\psi_{\pm} = c \log \lambda^2 + \Phi_{\pm} - \Phi . \tag{85}$$

The expression in (84) can be decomposed into a symmetric part given by (79) and an anti-symmetric part by (80) and the result is

$$\begin{aligned} u_r \cong |\tilde{x}\lambda|^{-1/2}\{&(\cos\psi_+ - A\sin\psi_+)\sin(\tilde{\Theta}+\Phi) \\ &+ \sin\psi_+[\cos(\tilde{\Theta}+\Phi) + A\sin(\tilde{\Theta}+\Phi)]\} . \end{aligned} \tag{86}$$

Now in order for $\tilde{u}_r$ to match (86) as $\tilde{x} \to +\infty$, we choose

$$\tilde{u}_r = x^{-1/2}[(\cos\psi_+ - A\sin\psi_+)u_s + (\sin\psi_+)u_a] . \tag{87}$$

It then follows from (79) and (80) that as $\tilde{x} \to -\infty$,

$$\begin{aligned} \tilde{u}_r = |\tilde{x}\lambda|^{-1/2}[&(\cos\psi_+ - 2A\sin\psi_+)\sin(\tilde{\Theta}+\Phi) \\ &-(\sin\psi_+)\cos(\tilde{\Theta}+\Phi)] . \end{aligned} \tag{88}$$

which must match (82). To see that this is possible, we rewrite (82) in terms of $\tilde{\Theta}$ as

$$u_r = G|\tilde{x}\lambda|^{-1/2}[\cos\psi_- \sin(\tilde{\Theta}+\Phi) + \sin\psi_- \cos(\tilde{\Theta}+\Phi)] \tag{89}$$

where ψ_- can be found in (85). Now in order for (88) and (89) to match, it is sufficient to have

$$\sin \psi_-(\cos \psi_+ - 2A \sin \psi_+) = -\sin \psi_+ \cos \psi_-$$

or, upon using (85),

$$\sin \Gamma - A[\cos \Gamma - \cos(\Phi_+ - \Phi_-)] = 0 \tag{90}$$

where Γ stands for

$$\begin{aligned} \Gamma &= \psi_+ + \psi_- \\ &= 2c \log \lambda^2 + \Phi_- + \Phi_+ - 2\Phi \, . \end{aligned} \tag{91}$$

The following argument shows that (90), as an equation for Γ, has infinitely many solutions. Let us assume that $\cos(\Phi_+ - \Phi_-) > 0$. We choose Γ_1 and Γ_2 such that

$$\cos \Gamma_{1,2} = \cos(\Phi_+ - \Phi_-)$$

and $0 < \Gamma_1 \leq \frac{\pi}{2}$ and $\frac{3}{2}\pi \leq \Gamma_2 < 2\pi$. Then, for $\Gamma = \Gamma_1$, $\sin \Gamma$ is positive and therefore the left hand side of (90) is positive, and for $\Gamma = \Gamma_2$, it is negative. Therefore eqn. (90) must have a root Γ_* in (Γ_1, Γ_2). If $\cos(\Phi_+ - \Phi_-) < 0$, we only have to consider $\frac{\pi}{2} \leq \Gamma_1 < \pi$ and $\pi < \Gamma_2 \leq \frac{3}{2}\pi$.

The general solution of (90) is then

$$\Gamma = \Gamma_* \pm 2n\pi$$

for n being any integer, and consequently

$$\lambda^2 = -\rho(r_o)\omega^2 E^{-2}$$
$$= \exp[c^{-1}(\Phi - \tfrac{1}{2}\Phi_+ - \tfrac{1}{2}\Phi_- + \tfrac{1}{2}\Gamma_* - n\pi)] \qquad (92)$$

and only large n will be taken so that ω is indeed small.

From (92) and (83), it is seen that there are infinitely many eigenvalues with $\omega = 0$ as an accumulation point. This is in agreement with the magneto-hydrodynamic result obtained by Grad[6]. On the other hand, if c is small, the maximum growth rate is bounded by

$$|\omega|^2 < [E^2/\rho(r_o)]\exp(-\Phi_*/c)$$

where Φ_* is obtained as the smallest possible positive value of $n\pi + \frac{1}{2}\Phi_+ + \frac{1}{2}\Phi_- - \frac{1}{2}\Gamma_* - \Phi$ for all possible integer n.

It is also noted from (87) that the amplitude of u_r is by a factor $\lambda^{-1/2}$ larger than the amplitude of the eigenfunction in the side intervals $(0, r_o - \varepsilon)$ and $(r_o + \varepsilon, R_o)$. This indicates that the eigenfunction is localized in the center layer around $r = r_o$, while the thickness of the layer is of the order λr_o. Finally, it is mentioned that the method we used in analyzing the eigenvalue problem does not make use of the detailed structure of the equation in the center layer and is therefore applicable to the magneto-hydrodynamic case equally well.

Acknowledgment

The author wishes to thank Professor H. Grad, Dr. A. Kadish and Dr. G. Spies for several helpful discussions. Thanks are also due to Professor H. Grad for introducing the author to this subject matter.

This research was jointly supported by the Division of Applied Mathematics, Brown University and by the U.S. A.E.C. under Contract AT(11-1)-3077 with the Magneto-Fluid Dynamics Division, Courant Institute of Mathematical Sciences, New York University.

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Part 2

LOW SHEAR INSTABILITIES AND GROWTH RATES

I. INTRODUCTION

In a recent work,[1] a normal mode stability analysis has been undertaken for a cylindrical diffuse pinch of guiding center plasma to study the kinetic effect on the stability criteria and the growth rates. It is shown that the problem can be reduced to a single second-order ordinary differential equation which may be designated as the stability equation. Complete knowledge of the stability of the pinch can only come from a thorough numerical analysis on the stability equation which should also give the growth rates associated with any possible instabilities. It is difficult to get general analytic results from this stability equation because of its complexity. For this reason it seems worthwhile to go into some special limiting situations where some analytic results can be drawn. The purpose of this work is to present some analytic results in the low shear limit.

The results are the line-preserving instabilities considered in Sec. II for zero shear pinches and the "semi-local" modes in Sec. III for low shear pinches. The threshold and growth rates for these instabilities are derived. It is shown that for low shear systems the large end of the "semi-local" spectrum joins the line-preserving spectrum. This shows that the line-preserving instabilities have higher growth rate than the "semi-local" modes and this justifies the zero-shear approximation for a low shear system as far as the maximum growth rate is concerned. It is also

shown that when Suydam-type instabilities occur, the Suydam spectrum joins the small end of the "semi-local" spectrum which indicates at least for the low shear system the Suydam instabilities have lower growth rate and are therefore not important. These results are in agreement with the corresponding results of the MHD theory.[2] A comparison between the present results and the MHD results is given in Sec. IV, where the difference between the guiding center theory and the MHD theory is discussed in terms of the threshold and growth rates of the "semi-local" and line-preserving instabilities.

II. SHEARLESS DIFFUSE PINCH

We first consider the case that the pinch is shearless, i.e. the rotational number

$$\mu = B_\phi/(rB_z) \tag{1}$$

is a constant. It is assumed that for the moment μ is equal to a rational number so that for some suitable k and m we have

$$H = kB_z + mB_\phi/r = 0 .$$

If we use the normal mode $\exp(ikz + im\phi - i\omega t)$, the stability equation may be written as

$$\frac{d}{dr}[F\frac{du}{dr}] + Gu = 0 \tag{2}$$

where u is the radial velocity component of the plasma and the coefficients F and G are given by

$$F = rK_1(w)NB^2[N-E^2K_1(w)] \tag{3}$$

$$N = \rho\omega^2 - \sigma^2H^2 \tag{4}$$

$$G = rC_o(w,r) \tag{5}$$

$$w = \omega B/H \tag{6}$$

and the functions K_1, E, σ^2 and C_o can be found in Ref. 1. It suffices to note here that if the equilibrium distribution functions are symmetric in the velocity v along the field lines, then the guiding center equations can be shown to

give rise to a symmetric operator.[3] In such case, we only have to consider ω to be purely imaginary for the unstable modes. If, furthermore, the equilibrium distribution functions are monotone decreasing in v^2, then it can be shown that $K_1(w)$ is real and positive. Consequently equation (2) is non-singular.

The boundary conditions for equation (2) are specified at $r = 0$ and $r = R$, the boundary of the plasma. At $r = 0$, we have

$$u = 0 \qquad (m \neq 0) \tag{7}$$

$$u = 1 \qquad (m = 0) \tag{8}$$

If the plasma is bounded by a solid conducting wall at $r = R$, the boundary condition is given by

$$u = 0 \qquad \text{at } r = R\ . \tag{9}$$

And if the plasma is separated from the solid wall by a vacuum region, then we have

$$a_1 u + a_2 \dot{u} = 0 \qquad \text{at } r = R \tag{10}$$

where a_1 and a_2 are given in Ref. 1.

For the case of shearless pinches, we set $H \equiv 0$ in F and G and taking the limit $w \to \infty$. The result is

$$F = r[1 + 2P_2 B^{-2}]\rho\gamma^2 B^2 [\rho\gamma^2 + E^2(1+2P_2 B^{-2})] \tag{11}$$

$$G = \rho\gamma^2 r - IB_\phi^2 B^{-2} \tag{12}$$

$$I = \dot{P}_1 + \dot{P}_2(1+P_2B^{-2}) + 2P_1B_\phi^2 r^{-1}B^{-2} + \sigma^2 B_\phi^2 r^{-1}[P_2B^{-2}+(2P_2+P_1)(B^2+2P_2)^{-1}] \quad (13)$$

where $\gamma = i\omega$ and the dot denotes differentiation with respect to r.

It then follows that for a shearless pinch

$$I \geq 0 \quad (14)$$

is a necessary condition for stability. To substantiate this claim, we only have to consider small values of γ for which F and G can further be approximated as

$$F = r\rho\gamma^2/\mathcal{K}^2 \quad (15)$$

$$G = -Ik^2/\mathcal{K}^2 \quad (16)$$

where $\mathcal{K}^2 = k^2 + m^2/r^2$, and let us assume that

$$I < 0$$

in a subinterval (r_1,r_2) and $I > 0$ in (o,r_1) and (r_2,R). We write equation (2) as

$$\frac{d}{dr}\left[\rho r^2 \frac{du}{dr}\right] - \lambda^2 I(B_\phi/B)^2 u = 0 \quad (17)$$

where $\lambda = 1/\gamma$. Since λ is large when γ is small, (17) can be solved by using the WKB approximation. Let us introduce the following variables

$$x_+ = \lambda \int_{r_1}^{r} |\xi(t)|^{\frac{1}{2}} dt \ , \quad (18)$$

$$x_- = \lambda \int_{r}^{r_2} |\xi(t)|^{\frac{1}{2}} dt \ , \quad (19)$$

$$\xi(r) = I\, k^2/(\rho r) \tag{20}$$

and first consider the simple case where the boundary conditions are given by $u(o) = u(R) = 0$. In this case, we require that $u(r)$ decays exponentially as $r \to 0$ and $r \to R$ so that the boundary conditions $u(o) = u(R) = 0$ are satisfied with an inaccuracy which is exponentially small in λ. Now, using the WKB connection formulas, we have (1) to the left of r_1:

$$u \cong a \cos(x_+ - \frac{\pi}{4}) \;, \tag{21}$$

(2) to the right of r_2:

$$u \cong a \cos(x_- - \frac{\pi}{4}) \;. \tag{22}$$

In order that (21) and (22) agree with each other, we must have

$$\lambda = (n + \frac{1}{2})\pi/Q \;, \tag{23}$$

where n is an integer and

$$Q = \int_{r_1}^{r_2} |\xi(t)|^{\frac{1}{2}} dt \tag{24}$$

It then follows that the growth rate is given by

$$\gamma = Q/[(n+\frac{1}{2})\pi] \;, \tag{25}$$

which is small if n is taken to be large. This shows that in the case where $\mu = B_\phi/(rB_z)$ is a constant whose value is a rational number, if $I < 0$ for some r, then there exists an infinite number of unstable eigenvalues going to zero.

If the boundary condition at $r = R$ takes the form of (10), then (25) is slightly changed and the result is

$$\gamma = Q/[(n+\alpha)\pi] \tag{26}$$

where the constant α depends on the coefficients a_1 and a_2 in (10).

On the other hand if $\mu = B_\phi/(rB_z)$ is an irrational number, then we can always choose k and m such that H is small throughout (0,R). Then the foregoing WKB analysis can still be carried out provided we restrict ourself to values of γ such that $\rho\gamma^2 >> \sigma^2H^2$, while γ itself is kept small. This indicates that

$$I > 0$$

is also a necessary condition for stability when μ is irrational. In this case the unstable eigenvalues are bounded away from the origin since we have $\rho\gamma^2 >> \sigma^2H^2$.

We may characterize this class of unstable modes by noting that under the condition $H = 0$ and $\dot{\mu} = 0$, the perturbation of the magnetic field is aligned with the unperturbed magnetic field [see equations (47) and (49) of Ref. 1]. For this reason, these modes may be termed the line-preserving modes.

III. LOW SHEAR STABILITY

In the last section the shearless systems are considered. We shall, in this section, consider the limiting cases where the shear $\dot{\mu}$ is small, but not necessarily zero. We introduce a small parameter ε to express the smallness of the shear and write

$$H = mB_z(\mu + k/m) = \varepsilon h \tag{27}$$

where k and m are specially chosen to make H small. We again consider small growth rate by writing

$$\gamma = \varepsilon g . \tag{28}$$

In the limit $\varepsilon \to 0$, the coefficients F and G in equation (2) take the simple form

$$F = \varepsilon^2 r[\rho g^2 + \sigma^2 H^2]/\mathcal{K}^2 \tag{29}$$

$$G = -p(r,\gamma B/H)B_\phi^2/B^2 \tag{30}$$

and p is given by

$$p(r,\gamma B/H) = \dot{P}_1 + \dot{P}_2 + \sigma^2 B_\phi^2/r + M_3(w) + [\sigma^2/K_1(w)][M_2(w) - \sigma^2 B_\phi^2] \tag{31}$$

$$M_2(w) = B_\phi^2 w^2 \vartheta_2(w)/(rB) \tag{32}$$

$$M_3(w) = -w^2\dot{B}\vartheta_2(w) - B_\phi^2[\rho w^2 - w^4\vartheta_3(w)](rB^2)^{-1} \tag{33}$$

with the functions $\vartheta_2(w)$ and $\vartheta_3(w)$ given in Ref. 1 [equations (36) and (37)].

Since F in (29) contains a small parameter ε^2, equation (2) can again be solved by the WKB approximation in a way similar to the analysis in Sec. II. The conclusion is that

a necessary condition for low shear stability is that

$$p(r,\gamma B/H) \geq 0 \tag{34}$$

for all possible γ at all r, where $p(r,\gamma B/H)$ is given by (31). To substantiate this, we assume that $p(r,\gamma B/H)$ is negative at some r for $\gamma = \gamma_o = \varepsilon g_o$ and we consider values of γ in a neighborhood of γ_o, i.e. $\gamma = \varepsilon g = \varepsilon(g_o+\delta)$. For δ sufficiently small, we have

$$p(r,gB/h) < 0$$

in a subinterval (r_1,r_2) where $r_i = r_i(\delta)$, $i=1,2$. Then according to the lowest order WKB approximation, the appropriate boundary conditions at r=0 and r=R can be satisfied if

$$\varepsilon^{-1}L(\delta) = n\pi + \alpha \tag{35}$$

for $n = 0,\pm1,\pm2,...$ where α is the same constant used in (26) and $L(\delta)$ is given by

$$L(\delta) = \int_{r_1}^{r_2} \left| p(r,gB/h)k^2/[(\sigma^2h^2+\rho g^2)r] \right|^{\frac{1}{2}} dr \tag{36}$$

where it is recalled that $g = g_o+\delta$ and $r_1 = r_1(\delta)$, $r_2 = r_2(\delta)$.

For sufficiently small δ, then (35) can be written as

$$\delta L'(0) = \varepsilon\phi(n,\varepsilon) \tag{37}$$

$$\phi(n,\varepsilon) = n\pi + \alpha - \frac{1}{\varepsilon}L(0) \tag{38}$$

where $L'(0)$ is the derivative of $L(\delta)$ with respect to δ at $\delta = 0$. Denoting by $n_o = n_o(\varepsilon)$ the integer which satisfies

$$0 < \phi(n_o,\varepsilon) < \pi \tag{39}$$

We write (37) as

$$\delta = \varepsilon[\phi(n_o,\varepsilon) + \ell\pi] \tag{40}$$

which determines δ as $\ell=0,\pm1,\pm2,\ldots$. It is noted that $\varepsilon\ell$ must be small so as to keep δ small as it is assumed.

This shows that if for some $\gamma_o > 0$ we have $p(r,\gamma_o B/H) < 0$ at some r, then it is always possible to choose a γ which differs from γ_o in an amount of the order ε^2 and which is given by

$$\gamma = \varepsilon(g_o+\delta)$$

with δ given by (40) such that the boundary conditions at $r=0$ and $r=R$ are satisfied. Consequently, the system is unstable. Furthermore, it follows from (40) that the spacing of the unstable eigenvalues is of order ε^2. These instabilities correspond to the "semi-local" modes in the MHD diffuse pinch.[2] For these instabilities the maximum growth rate is given by

$$\gamma_{Max} = \underset{0\le r\le R}{\mathrm{Max}} \{\gamma:\ p(r,\gamma B/H) < 0\} \tag{41}$$

with an error of ε^2-order.

It is worthwhile to note that as $\gamma B/H \to \infty$,

$$\lim p(r,\gamma B/H) = I \tag{42}$$

where I is given by expression (13). This seems to suggest that if $I < 0$ at some r then expression (40) would imply that the maximum growth rate is unbounded. What this actually means is that the growth rate γ is no longer of order ε when I becomes negative. To see this, we return to (37) and consider ε fixed and let $g_o = \gamma_o/\varepsilon$ become large. Then the coefficient $L'(0)$ of δ becomes increasingly small since

$$L'(0) \sim 1/g_o ,$$

which shows that in order for δ to remain small g_o must remain much smaller than $1/\varepsilon$. In fact, for large γ (i.e. $\rho\gamma^2 >> \sigma^2H^2$), we can neglect σ^2H^2. This will lead us back to the result of Sec. II and the resultant growth rate is then given by (25). In other words, when I becomes negative, the large end of the closely spaced spectrum of the "semi-local" modes joins the spectrum of the line-preserving modes examined in Sec. II.

On the other hand as $\gamma B/H \to 0$, we have

$$p(r,o) = \dot{P}_1 + \dot{P}_2 + \sigma^2(B_\phi^2/r)[1 - \sigma^2/K_1(o)] .$$

Now if $p(r,o) < 0$ at some r, then there exist unstable semi-local modes whose eigenvalues are given by (40) as

$$\gamma = \gamma_o + \varepsilon\delta = \varepsilon\delta$$

where δ can be calculated from (37) and must remain positive. This shows that γ is bounded away from the origin by a distance of order ε^2. Now if the guiding center Suydam condition[1] is violated; i.e. if $H = 0$ at $r = r_o$ and if

$$S = p(r,o) + \frac{1}{4}\sigma^2 B_z^2 (\dot{\mu}/\mu)^2 < 0$$

at $r = r_o$, then there appear infinitely many Suydam modes whose eigenvalues go to zero with the origin as an accumulation point. This shows that when the low shear system is Suydam unstable the small end of the semi-local spectrum joins the Suydam spectrum.

IV. CONCLUDING REMARKS

In the preceding sections we have studied the line-preserving instabilities in a zero-shear diffuse pinch and the semi-local instabilities in a low shear diffuse pinch. It is shown that the large end of the semi-local spectrum joins the line-preserving spectrum and the small end joins the Suydam spectrum. This is in agreement with the MHD result of Grad.[2] This shows that the zero-shear approximation is a good model for low shear systems in finding out the maximum growth rate given by the line-preserving unstable modes.

This suggests that a possible estimate for the maximum growth rate for a low shear diffuse pinch may be obtained from (26) as

$$\gamma_{Max} \sim \frac{2}{\pi} \int_{r_1}^{r_2} |k^2 I/(pr)|^{\frac{1}{2}} dr \qquad (45)$$

Since I, given in (13), contains only P_1, P_2 and $\underset{\sim}{B}$, the threshold and the growth rate of the line-preserving instability do not depend on the details of the distribution functions. In the scalar pressure case we have from (13)

$$I = \dot{P}(2 + P/B^2) + 6(B_\phi^2/r)(P/B^2)(B^2+P)/(B^2+2P) \qquad (46)$$

while the corresponding MHD theory gives

$$I = 2\dot{P} + 4(B_\phi^2/r)\rho a^2/(B^2 + \rho a^2) \qquad (47)$$

where $a^2 = \partial P/\partial\rho$; i.e. a is the sound speed. For low-β plasma (46) and (47) can be made to agree by taking the ratio

of specific heats $C_p/C_v = 3/2$. But in high-β plasma no such simple relation exists between (46) and (47).

In the cases of scalar pressure where B_ϕ is much smaller than the longitudinal field component B_z, the Suydam instability condition in (44) implies the line-preserving instability $I < 0$ where I is given by (46). This reinforces the importance of the line-preserving instability over the Suydam instability for low shear system. In other words, if a low shear system with $B_\phi^2 \ll B_z^2$ is Suydam unstable, then it has line-preserving unstable modes which give the maximum growth rate.

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